KINETIC ENERGY FLUX *VS* POYNTING FLUX IN MHD WINDS AND JETS: THE INTERMEDIATE REGIME

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ABSTRACT

We show that the formal asymptotic limit for all rotating polytropic axisymmetric perfect MHD flows is a kinetic energy dominated wind which collimates to paraboloids around the symmetry axis. We reach this result by showing that another, *a priori* possible, solution with finite Poynting flux can be excluded on the following physical grounds: (1) the wind velocity does not exceed the fast mode speed everywhere and (2) the circumpolar current increases with distance from the source.

We show that asymptotic hoop stress collimation is mathematically robust and we give strong arguments why recent 'anti-collimation' claims are not correct.

However, in practice, due to the very slow logarithmic decline of the circumpolar current with increasing distance from the source, there is a broad intermediate regime with significant Poynting flux. This intermediate regime, rather

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than the mathematically exact asymptotic regime, may well apply along the finite length of the jet. We briefly discuss peculiarities that would be associated with Poynting jets in the intermediate regime. Force-free initial conditions in the near field are most likely to produce such jets, in which most of the energy flux is electromagnetic.

1. Introduction

We have previously established (Heyvaerts and Norman (1989)) that any stationary polytropic axisymmetric magnetized wind will collimate to paraboloids or cylinders along the symmetry axis at large distances from the source according to whether the electric current brought to infinity per hemisphere by the wind is finite or vanishes. In two companion papers (Heyvaerts and Norman 2002a,b) we have presented explicit asymptotically-matched solutions for both classical and relativistic winds. Our results however have left open the question of whether the asymptotic circumpolar current vanishes or not. We now address this issue which we already touched in earlier reports on our work (Heyvaerts and Norman (1997), Heyvaerts (1999)).

Previous work has described the transfield equation (Okamoto (1975) Heinemann and Olbert (1978)) that expresses the balance of forces in a general magnetized rotating flow in the presence of a gravitational field. Blandford and Payne (1982) showed how collimated flows from disks could be obtained with a similarity solution associated with a particular scale free current and field distribution on a Keplerian disk. Contopoulos and Lovelace (1994) have found a similarity solution incorporating gravity and pressure, extending and generalizing the Blandford and Payne approach, as also did Ostriker (1997). Lovelace and collaborators have also considered relativistic jets, using the force free approximation (Lovelace (1976), Lovelace et al. (1991), Lovelace et al. (1993)). In Lovelace et al. (2002) they find collimated Poynting flux jets with surrounding winds. Numerical solutions of the steady state rotating axisymmetric MHD system have been presented by Sakurai (1985) and Sakurai (1987) for the split monopole and magnetized disk. A recent study of the relativistic case is given by Bogovalov (2001). Collimation along the axis is clearly evident in both cases. This result has been strengthened by a number of analytical and numerical studies (Sauty et al. (1992a), Sauty et al. (1992b), Sauty and Tsinganos (1994)). Shu and collaborators, have analysed protostellar outflows (Shu et al. (1994a), Shu et al. (1994b), Najita and Shu (1994), Shu et al. (1995)). Pelletier and Pudritz (1992) produced special solutions that exhibited focussing and recollimation. Numerical results (Ustyugova et al. (1995), Ustyugova et al. (2000), Ouyed and Pudritz (1997a), Ouyed and Pudritz (1997b), Krasnopolsky et al. (1999))

further confirmed this.

Heyvaerts and Norman (1989) have used asymptotic analysis to understand the general properties of the shape of the field lines and the flow structure at large distances from the source as a function of the conserved flow quantities. They have shown that the structure of the field and flow at infinity is controlled by the total poloidal electric current flowing in the wind about the polar axis. In the case where this current vanishes, the poloidal field (and flow) lines asymptotically approach to parabolae which focus to the polar axis in such a way that both $r \rightarrow \infty$ and $(z/r) \rightarrow \infty$ where standard cylindrical coordinates (r, θ, z) are used with z along the symmetry axis. An expression giving the shape of these lines at infinity far from the polar axis, as a function of the flux function was derived. When the circum-polar poloidal current is non-zero, the poloidal field (and flow) surfaces have been shown (Heyvaerts and Norman (1989), Heyvaerts and Norman (1996)) to asymptotically approach to cylinders (possibly nested in asymptotically conical flux surfaces). The asymptotic radii of these cylindrical flux surfaces are given as a function of magnetic flux in the super-Alfvénic regions. In a relativistic generalization of our 1989 paper, Chiueh, Li and Begelman (1990) showed our results held not only for the non-relativistic case but also for the relativistic case.

In a recent paper Okamoto (2003) insists that the asymptotic circum-polar current must vanish and that the asymptotic structure of magnetic surfaces must be of the conical type. Our earlier results (Heyvaerts and Norman 1989) by no means imply that the asymptotic circum-polar current should assume a non-vanishing value, contrary to what Okamoto (2003) claims. This point has been made clearly in a number of conference reports (Heyvaerts and Norman (1997), Heyvaerts (1999)). On the contrary, our present results, which extend and make precise previous arguments published in these reports, just conclude the contrary, i.e., that the circumpolar current vanishes aymptotically in the mathematical sense. Thus, there is no disagreement between us on this point. By contrast, Okamoto's antifocusing statement is inconsistent with an asymptotically vanishing circumpolar current. A non focusing magnetic surface must obviously be a surface such that (z/r) approaches a finite value at infinite distance from the origin on that surface, or, if such a limit does not exist, it is a surface along which (z/r) remains bounded from above. On such a magnetic surface ρr^2 must approach a finite value (or remain bounded from below) at infinite distance from the origin. Indeed, an elementary calculation shows that when (z/r) approaches a finite limit (or is bounded) so does $r|\vec{\nabla} a|$, which implies that the polodal current enclosed in this surface approaches a finite value (see Heyvaerts and Norman (1989) and Heyvaerts and Norman (1996) for details). If the mathematical asymptotic state, because of a very slow convergence, is to be reached only at distances larger than the actual length of the jet, there is of course no point to confront the actual wind structure with this asymptotic solution. While this is the point made in this paper, it is not what is meant by Okamoto (2003) when

he claims conical asymptotics.

The aim of this paper is precisely to determine whether, in general, the asymptotic circumpolar current vanishes or is finite. This question cannot be dealt with in the framework of a self-similar model, nor of any model which imposes a priori constraints on the solution. In this paper we discuss this issue in terms of the properties of the first integrals of the motion, which we take as given. These first integrals are determined by boundary conditions and by the criticality conditions. We consider both classical and relativistic winds.

The paper is structured as follows. In Section 2 we review the basics of both relativistic and non-relativistic stationnary axisymmetric rotating MHD winds. In section 3 we summarize results on the asymptotic structure of MHD winds relevant to the distribution of flux in the asymptotic domain. In section 4 we show that a necessary condition for solutions with a non-vanishing circumpolar asymptotic current to be possible is that the function $\alpha(a)E(a)/\Omega(a)$ has a minimum value, I_{sup}, away from the polar axis. We show that this non-vanishing asymptotic circumpolar poloidal electric current is proportional to this minimum value. The analysis extends to relativistic winds, for which a non-vanishing asymptotic proper current must be the minimum value, K_{sup}, of $\alpha(a)(E(a)-c^2)/\Omega(a)$. We next show in section 5 that classical winds which are super fast mode at infinity must have a current less than $\frac{2}{3}I_{sup}$. A similar upper bound is derived for relativistic winds. Due to the very slow decline of the current, the final asymptotic regime may in practice not be reached, allowing for intermediate quasi-asymptotic regimes. These are discussed in section 6 where we emphasize systems carrying close to the maximum current. These winds split into a separate conical jet of small opening angle and a circum-equatorial wind separated by a region where the energy flux is almost all in Poynting form. We summarize our results in section 7.

2. General Properties of MHD Winds

2.1. Notation and Definitions

We study perfect MHD, axisymmetric, stationary, polytropic jets and winds. The physical quantities are denoted by their usual symbols, ρ, p, $\vec{v}$, $\vec{B}$ for the density, pressure, velocity and magnetic field respectively. The gravitational potential is Φ_G. Any axisymmetric vector quantity can be split into poloidal and toroidal components. The magnetic flux Φ_m through a circle of radius r centered on the axis at altitude z is written as $\Phi_m = 2\pi a(r,z)$. The poloidal part of the magnetic field can then be expressed as

$$\vec{B}_P = -\frac{1}{r}\,\frac{\partial a}{\partial z}\vec{e}_r + \frac{1}{r}\,\frac{\partial a}{\partial r}\vec{e}_z \tag{1}$$

The magnetic surfaces, generated by the rotation of field lines about the axis, are surfaces of constant $a(r, z)$. The value of a suitably labels them.

2.2. First Integrals

A rotating, axisymmetric, stationary, polytropic perfect MHD wind flow admits five first integrals of the motion (see for example Heyvaerts (1996)) α, Ω, L, E and Q which are conserved on magnetic surfaces, $a(r, z)$. For classical winds, these first integrals are defined by:

$$\rho \vec{v}_P = \alpha(a) \vec{B}_P, \tag{2}$$

$$\rho v_\theta = \alpha(a) B_\theta + \rho r \Omega(a) \tag{3}$$

$$L(a) = r v_\theta - \frac{r B_\theta}{\mu_0 \alpha(a)}. \tag{4}$$

$$E(a) = \frac{1}{2} v_P^2 + \frac{1}{2} v_\theta^2 + \frac{\Gamma}{\Gamma - 1} \frac{p}{\rho} + \Phi_G - \frac{r B_\theta \Omega(a)}{\mu_0 \alpha(a)} \tag{5}$$

$$p = Q(a) \rho^\Gamma \tag{6}$$

Γ is the polytropic index. The Alfven radius $r_A(a)$ and the Alfven density $\rho_A(a)$ are defined by

$$r_A^2(a) = L(a)/\Omega(a) \qquad\qquad \rho_A(a) = \mu_0 \alpha^2(a) \tag{7}$$

The Alfvén point on a magnetic surface a is at $r = r_A(a)$. The density at this point is ρ_A. The toroidal variables v_θ and B_θ can be expressed in terms of first integrals and of the density ρ as:

$$v_\theta = \frac{L}{r} + \frac{\rho}{r} \frac{L - r^2 \Omega}{\mu_0 \alpha^2 - \rho}, \tag{8}$$

$$B_\theta = \frac{\mu_0 \alpha \rho}{r} \frac{L - r^2 \Omega}{\mu_0 \alpha^2 - \rho}. \tag{9}$$

2.3. Relativistic Winds

Relativistic flows are characterized by a local Lorentz factor, γ, defined by:

$$\gamma = \left(1 - \frac{(v_\theta^2 + v_P^2)}{c^2} \right)^{-1/2} \tag{10}$$

The proper rest mass density is denoted by ρ and the specific momentum (the momentum per unit mass) is

$$\vec{u} = \gamma\vec{v} \tag{11}$$

The proper gas pressure is assumed to be related to the proper density by Eq.(6), where Q is constant following the fluid motion. We define the function

$$\xi = 1 + \frac{\Gamma}{\Gamma - 1}\frac{Q\rho^{\Gamma-1}}{c^2}. \tag{12}$$

which is also equal to $(1 + \int dP/\rho c^2)$ calculated at constant entropy for a polytropic gas. We denote the electric charge density by ρ_e, the electric current density by $\vec{j}$ and the mass of the central object by M_*. The special-relativistic equation of motion can be written as:

$$\gamma\rho(\vec{v}\cdot\vec{\nabla})(\gamma\xi\vec{v}) = -\vec{\nabla}P + \vec{j}\times\vec{B} + \rho_e\vec{E} + \gamma\rho\vec{\nabla}\left(\gamma\xi\frac{GM_*}{R}\right) \tag{13}$$

The relativistic form of the laws of mass conservation, isorotation, angular momentum conservation and Bernoulli involve surface functions E, α, L, Ω and Q, which are in this case defined by the relations:

$$\gamma\rho\vec{v}_P = \alpha(a)\vec{B}_P, \tag{14}$$

$$\gamma(v_\theta - r\Omega(a)) = \alpha(a)B_\theta/\rho, \tag{15}$$

$$\gamma\xi r v_\theta - \frac{rB_\theta}{\mu_0\alpha(a)} = L(a), \tag{16}$$

$$\gamma\xi(c^2 - \frac{GM_*}{R}) - \frac{r\Omega(a)B_\theta}{\mu_0\alpha(a)} = E(a) \tag{17}$$

Note that E in Eq.(17), now includes the rest mass energy. The rotation rate of the magnetic field, $\Omega(a)$, which appears in equations (15) and (17), is defined in terms of the electric field by:

$$\vec{E} = -\Omega(a)\vec{\nabla}a \tag{18}$$

Toroidal variables may be expressed using Eqs.(16) and (15):

$$rB_\theta = \mu_0\alpha\rho\ \ \frac{L - \gamma r^2\xi\Omega}{\mu_0\alpha^2\xi - \rho} \tag{19}$$

$$\gamma\xi v_\theta = \frac{L}{r} + \frac{\rho}{r}\,\frac{L - \gamma r^2\xi\Omega}{\mu_0\alpha^2\xi - \rho} \tag{20}$$

Since γ depends on v_θ, the elimination of the toroidal variables in Eqs.(19) and (20) is not yet complete. These expressions can be substituted in the Bernoulli equation (17), which

can then be solved to obtain an expression of $\gamma\xi$ in terms of the poloidal variables. This eventually gives the toroidal variables in terms of poloidal variables alone as:

$$rB_\theta = -\mu_0\alpha \frac{L(c^2 - GM_*/R) - r^2\Omega E}{(c^2 - GM_*/R)(1 - \mu_0\alpha^2\xi/\rho) - r^2\Omega^2} \tag{21}$$

$$rv_\theta = r^2\Omega \left(1 \quad - \frac{\mu_0\alpha^2\xi}{\rho} \frac{L(c^2 - GM_*/R) - r^2\Omega E}{(1 - \mu_0\alpha^2\xi/\rho) r^2\Omega E - r^2\Omega L}\right) \tag{22}$$

These expressions have to be regular where their denominator vanishes, defining quantities which play the role of Alfvén density and Alfvén radius in the relativistic context.

3. Asymptotics of MHD Winds

3.1. Semantics

We define important expressions to be used in this paper. The *asymptotic domain* consist of all points which are located on their own magnetic surface far away from the corresponding Alfvén point. A magnetic surface is said to be *asymptotically parabolic* if the limits following this surface of r and (z/r) are both infinite. A magnetic surface is said to be *asymptotically conical* if r approches infinity and (z/r) approaches a finite limit, $\tan\psi_\infty(a)$. This does not imply that $(z - r\tan\psi_\infty)$ approaches a finite limit: conical magnetic surfaces may have parabolic branches. An asymptotically cylindrical magnetic surface is one on which r approaches a finite value, $r_\infty(a)$. The ratio r/r_A then also approaches a finite limit. When r becomes infinite on a magnetic surface, the surface is said to flare out.

A *neutral* or *null* magnetic surface is one on which the poloidal field vanishes. The toroidal field also vanishes, since it is generated from the poloidal field by rotation. A neutral surface is strictly speaking not a magnetic surface, since it contains no magnetic field lines. Null surfaces are sandwiched between regular magnetic surfaces, so that the definition still retains meaning in the limit. The immediate vicinity of neutral magnetic surfaces is of special interest since they are exceptional regions where electric currents can flow in the asymptotic domain (Heyvaerts and Norman (2002a), Heyvaerts and Norman (2002b)). We refer to these regions as neutral surface boundary layers. The polar boundary layer in the vicinity of the polar axis is similarly a region where electric currents can flow in the asymptotic domain. We call the region away from boundary layers the field region.

The total poloidal current, $J = 2\pi rB_\theta/\mu_0$, is the electric current through a circle of axis z passing the point (r, z). For positive Ω and α, J is negative. We shall loosely refer to the quantity

$$I = -rB_\theta/\mu_0 \tag{23}$$

as the current. The total poloidal electric current enclosed in a neutral surface is zero. This means that neutral surfaces separate the wind in a number of cells in each of which the total current separately closes. We refer to the space between two neighbouring neutral magnetic surfaces as being a cell. In the asymptotic domain, current-carrying regions are restricted to regions of small extent (Heyvaerts and Norman (2002a),Heyvaerts and Norman (2002b)).

There is a total current that flows about the polar axis which we refer to as the *circumpolar current*. This circumpolar current varies with the distance to the wind source.

3.2. Relevant Results

For $r \gg r_A$, the azimuthal velocity vanishes while the current approaches the value

$$I \approx \frac{\rho r^2 \Omega(a)}{\mu_0 \alpha(a)} \tag{24}$$

Heyvaerts and Norman (1989) show that ρr^2 is bounded from above. The toroidal component of the velocity, v_θ, approaches zero as r approaches infinity on any flaring magnetic surface. Thus the flow velocity becomes poloidal while I remains bounded. If (ρr^2) approaches a finite limit, the asymptotic structure of the magnetic surfaces consists of a set of conical surfaces, inside of which cylindrical magnetic surfaces are nested. Such flows convey Poynting flux to infinity. On cylindrical magnetic surfaces none of these results strictly apply. Here, as in our previous papers, we nevertheless assume that the asymptotic cylindrical radius on any magnetic surface is much larger than the corresponding Alfvén radius. If ρr^2 does converge to zero at large distances, the asymptotic structure of magnetic surfaces consists of nested paraboloids (Heyvaerts and Norman 1989). No Poynting flux reaches infinity and all the energy emerges in kinetic form.

A detailed asymptotic solution in terms of given first-integrals has been presented in the companion papers (Heyvaerts and Norman 2002a,b) for both classical and relativistic winds. We have shown that electric currents are distributed in the asymptotic domain in boundary layers about the pole and in the vicinity of neutral magnetic surfaces, with very little current density flowing outside these regions. Detailed solutions for the inner structure of these current-carrying boundary layers were also discussed. Let a_n be the flux variable of some neutral magnetic surface. If there is a mass flux along the surface, the first integral α diverges for $a \to a_n$, as can be seen from Eq.(2). We have established that outside these boundary layers the transfield equation takes the simple form

$$I = I_\infty(b) \tag{25}$$

where b labels surfaces orthogonal to magnetic surfaces. Specifically, the label b is taken to be the value of the z-coordinate on this orthogonal trajectory at the polar axis. Variables a and b could, in principle, be taken as space coordinates replacing r and z, with a playing the role of an angular variable and b the role of a radial variable.

The current variable, I, in general depends on both a and b. Eq.(25) expresses the fact that its dependance on a almost completely disappears in the field-regions of the asymptotic domain, outside the current carrying boundary layers. More precisely, $I_\infty(b)$ is only piecewise constant, reversing sign when crossing from one cell to another. This follows from consideration of the equilibrium of the neutral sheet boundary layers where the dominant toroidal magnetic pressure on each side must balance the other. So, B_θ, and I reverse sign as neutral magnetic surfaces are crossed. This means that the absolute value of the total poloidal current $|I_\infty(b)|$ passes unchanged from one to the next cell.

Winds therefore could come in two separate classes: Poynting jets approaching a finite value of $|I_\infty(b)|$ or kinetic winds having asymptotically vanishing current. Whether both classes actually occur in nature remains to be discussed.

Outside the current-carrying boundary layers, the Lorentz force $\vec{j} \times \vec{B}$ vanishes since $\vec{j}_P$ and $\vec{B}_P$ become both small. Inside boundary layers, other forces balance the Lorentz force (Sauty et al. 1999). The gas pressure is the most likely additional force in the case of polytropic flows since its decline with r is the slowest. The polar boundary layer has the structure of a column pinch, while the neutral surface boundary layers have the structure of sheet pinches (Heyvaerts and Norman (2002a)).

3.3. Distribution of Flux on Orthogonal Trajectories: Classical Winds

For large values of r/r_A, it is possible to determine the distribution of flux along trajectories orthogonal to magnetic surfaces by quadrature. Denoting the terminal velocity on surface a by $v_\infty(a)$, equations (2), (5) and (9) simplify to the form:

$$\rho r v_\infty(a) = \alpha |\vec{\nabla} a| \tag{26}$$

$$\mu_0 I = + \frac{\rho r^2 \Omega}{\alpha} \tag{27}$$

$$E = \frac{v_\infty^2}{2} + \frac{I \Omega}{\alpha} \tag{28}$$

Eliminating v_∞ and ρ we obtain an expression for $r|\vec{\nabla} a|$ in terms of the first integrals and I:

$$r|\vec{\nabla} a| = \frac{\sqrt{2}\mu_0 |I|}{\Omega} \left(E - \frac{I\Omega}{\alpha} \right)^{\frac{1}{2}} \tag{29}$$

Let the curvilinear abcissa along the trajectories orthogonal to the magnetic surfaces be denoted by σ, conventionally increasing from pole to equator. Then $|\vec{\nabla} a| = da/d\sigma$, and the relation (29) becomes:

$$\frac{d\sigma}{r} = \frac{\Omega(a)da}{\mu_0 I(a,b)\sqrt{2}\sqrt{E - I(a,b)\Omega(a)/\alpha(a)}} \tag{30}$$

This relation holds true both in the field-regions, where $I(a, b)$ does not depend on a, and in current carrying boundary layers, where this dependence is rather strong.

The actual form of the quadrature relation (30) depends on the actual shape of the orthogonal trajectories to magnetic surfaces. These trajectories have been shown to be approximately circles when $I(a, b)$ approaches a non-vanishing limit I_∞ (Heyvaerts and Norman (2002a), Heyvaerts and Norman (2002b)). This result has been extended in a WKB sense to the case when I slowly varies with b. Asymptotically cylindrical surfaces are approximately orthogonal to circles too. The position on such a circular orthogonal trajectory is specified by a latitude angle ψ from the equator, so that

$$z = r \ \tan(\psi(a)) \tag{31}$$

For approximately circular orthogonal trajectories Eq.(30) integrates to

$$\tan(\psi(a)) = \sinh\left(\int_a^{a_1} \frac{1}{\sqrt{2}\mu_0 I(a,b)} \frac{\Omega(a')da'}{\sqrt{E(a') - I(a,b)\Omega(a')/\alpha(a')}}\right) + \tan(\psi(a_1)) \tag{32}$$

where a_1 is some reference flux. Along the orthogonal trajectory b the current $I(a, b)$ vanishes near the polar axis proportionally to a and causes $\tan(\psi(a, b))$ to diverge as a approaches zero. Outside boundary layers, $I(a, b)$ becomes a function $I_\infty(b)$, essentially independent of a at constant b, and the solution (32) simplifies. It will be shown below that the integral in Eq.(32) converges at a neutral surface. If, moreover, we neglect the flux in neutral boundary layers the integration in Eq.(32) can be extended from the equator, at flux $a = A$, where $\tan\psi(A) = 0$, to the surface a with a constant value of $|I(a, b)| = I_\infty$. This gives an approximate expression for $\psi(a)$:

$$\tan(\psi(a)) = \sinh\left(\int_a^{A} \frac{1}{\sqrt{2}\mu_0 I_\infty(b)} \ \frac{\Omega(a')da'}{\sqrt{E(a') - I_\infty(b)\Omega(a')/\alpha(a')}}\right) \tag{33}$$

3.4. Distribution of Flux on Orthogonal Trajectories: Relativistic Winds.

A similar analysis can be done in the case of relativistic winds (Heyvaerts and Norman (2002b)). In the field-regions the quantity K, defined by:

$$K = \frac{r}{\mu_0 c} \sqrt{c^2 B_\theta^2 - \Omega^2 |\vec{\nabla} a|^2} \tag{34}$$

becomes a constant. We refer to K as the proper total current (Heyvaerts and Norman 2002b). K is r times the asymptotic value of the electromagnetic invariant $(B^2 - E^2/c^2)^{1/2}$. It reduces in the classical limit to the poloidal current I (see Eq.(50)). In field-regions K becomes independent of a along any orthogonal trajectory, giving:

$$K = K(b) \tag{35}$$

The asymptotic relativistic Bernoulli equation can be written as:

$$u_{P\infty} \equiv \gamma_\infty v_{P\infty} = c \frac{\sqrt{E^2 - (c^2 + K\Omega/\alpha)^2}}{(c^2 + K\Omega/\alpha)} \tag{36}$$

Note that in relativistic dynamics E contains the rest-mass specific energy, c^2, so that $E_{class} = E_{relat} - c^2$. The relativistic form of Eq.(30) is:

$$\frac{d\sigma}{r} = \frac{\Omega(a) \sqrt{c^2 - v_\infty^2(a)}\ da}{\mu_0 c K(a,b)\ v_\infty(a)} \tag{37}$$

For relativistic winds, as for classical ones, the orthogonal trajectories to magnetic surfaces at large distances closely resemble circles, in which case Eq.(37) can be integrated to find the latitude $\psi(a)$ of the magnetic surface a on the orthogonal trajectory b. This gives:

$$\tan\left(\psi(a,b)\right) = \sinh\left(\int_a^{a_1} \frac{1}{\mu_0 c\ K(a,b)} \frac{\Omega(a')da'(c^2 + K(a,b)\Omega(a')/\alpha(a'))}{\sqrt{E^2(a') - (c^2 + K(a,b)\Omega(a')/\alpha(a'))^2}} \right) + \tan\psi(a_1, b) \tag{38}$$

where a_1 is a reference flux in the same cell as a itself. In the case of dipolar symmetry, only one cell would be present per hemisphere. Neglecting the flux in the equatorial boundary layer, a_1 is the equatorial flux, A, and $\tan\psi(A,b)$ vanishes. Outside boundary layers, $K(a,b)$ becomes a constant $K_\infty(b)$, independent of a and the solution given by Eq.(38) simplifies as:

$$\tan\left(\psi(a,b)\right) = \sinh\left(\int_a^{A} \frac{1}{\mu_0 c\ K_\infty(b)} \frac{\Omega(a')da'(c^2 + K_\infty(b)\Omega(a')/\alpha(a'))}{\sqrt{E^2(a') - (c^2 + K_\infty(b)\Omega(a')/\alpha(a'))^2}} \right) \tag{39}$$

4. The Circumpolar Current

4.1. Mixed Cylindrical-Conical Asymptotics

When $I(a,b)$ converges to zero the solution, which consists of nested paraboloids, fills all space. We have shown in Heyvaerts and Norman (2002a) that, conversely, if $|I(a,b)|$ approaches a non-vanishing value at infinity, the asymptotic structure of magnetic surfaces consists of cylindrical surfaces, possibly nested in conical ones. These two geometries seem to clash because their common existence implies the presence of a big spatial region with no magnetic surface. between the last cylindrical surface and the first conical surfaces. A smooth connection is nevertheless possible. Let a_* be the flux at which the magnetic surfaces switch from being cylindrical to being conical. The solution is continuous if $r_\infty(a)$ approaches an infinite limit as $a \to a_*$ and if $\psi(a)$ approaches $\pi/2$, as a approaches a_* from above. Thus, the difficulty of matching cylindrical to conical asymptotics is not sufficient to eliminate the idea that the total circumpolar current could remain finite. The relevant question is rather whether the wind's physics allows this. In a cylindrical region, $r_\infty(a)$ cannot approach infinity uniformly as a approaches a_*. Indeed, the limiting magnetic surface has the character of a paraboloidal surface described by $z = F_*(r)$. Any asymptotically cylindrical magnetic surface with terminal radius $r_\infty(a)$ is nested inside $z = F_*(r_\infty(a))$. Since $F_*(r)$ diverges with r , the limit $r_\infty(a)$ can obviously not be reached uniformly. For future reference let us note that the terminal radius, $r_\infty(a)$ is given by:

$$r_\infty(a) = r_1 \exp\left(\int_{a_1}^{a} \frac{1}{\sqrt{2\mu_0 I_\infty}} \frac{\Omega(a')da'}{\sqrt{E(a') - I_\infty \Omega(a')/\alpha(a')}} \right) \tag{40}$$

4.2. Asymptotic Current for Classical Winds

Eqs.(32) and (38) give the flux distribution explicitly in the field regions. Separate solutions have been obtained in current-carrying boundary layers and matched to the field-region solutions. For classical winds, a smooth asymptotic matching implies a relation, which we refered to as the Bennet relation, between the current $I(b)$ and the axial density at the distance b, $\rho_0(b)$:

$$\frac{\Gamma}{\Gamma - 1}\ Q_0 \rho_0^{\Gamma-1}(b) = \frac{I(b)\ \Omega_0}{\alpha_0} \tag{41}$$

Matching also imposes the relation:

$$\frac{\lambda(n_0(b))}{n_0^{\Gamma-1}(b)} = (2-\Gamma)\ \ln(n_0(b))\ +\ \ln\left(\frac{4b^2}{\ell^2}\right) \tag{42}$$

Here $n_0(b) = \rho_0(b)/\rho_{A0}$ is a dimensionless measure of the axial density and ρ_{A0} is the Alfvén density on the polar field line. The reference length ℓ is defined by

$$\ell^2 = \frac{\Gamma}{\Gamma - 1} \frac{Q_0 \rho_{A0}^{\Gamma-1}}{\Omega_0^2} \tag{43}$$

and λ is the following integral, which depends on $I(b)$, that is, by Eq.(41), on $n_0(b)$ or $\rho_0(b)$

$$\lambda = \int_0^A \frac{\Omega(a')\ \sqrt{E_0}}{\Omega_0\ \sqrt{E(a') - I(b)\Omega(a')/\alpha(a')}}\ \frac{da'}{a_0} \tag{44}$$

As b approaches infinity, some term of Eq.(42) has to balance the divergence of the logarithmic term on the right. If $I(b)$ asymptotically vanishes, so does $\rho_0(b)$(Eq.(41)), and λ approaches a finite value. In this case, Eq.(42) is satisfied with $\rho_0(b)$ approaching zero, consistent with Eq.(41). If, by contrast, $I(b)$ approaches a finite value, I_∞, so does $n_0(b)$ (Eq.(41)). The divergence of the second logarithmic term on the right hand side of Eq.(42) can then only be matched by a divergence of $\lambda(n_0(b))$. As discussed in Heyvaerts and Norman (2002a), this implies that, as b approaches infinity, the circumpolar poloidal current $I(b)$ rises to its maximum allowed value. This value is the absolute minimum of the function $(\alpha E/\Omega)$, reached away from the polar axis at $a*$ (see Fig.1). Then:

$$|I_\infty| = I_{sup} = \mathrm{Min}_{a\neq 0}\ |\frac{\alpha(a)E(a)}{\Omega(a)}| \tag{45}$$

For this current both $\tan(\psi_\infty(a))$ and the terminal radius $r_\infty(a)$ would simultaneously diverge as $a*$ is approached.

From Eq.(40) we see that $|I|$ should in this case increase with b. However, this result gives a direct contradiction since the circumpolar current must close back to the source *via* a low current density flow in the field-region. This picture implies that $|I(b)|$ must approach its limit from above. A decreasing $|I(b)|$ accelerates the outflow by the coiled spring force associated with the gradient of the toroidal magnetic field pressure. A monotonically decreasing $|I(b)|$ corresponds to continued, though weakening, acceleration from the wind source to infinity. An increasing $|I(b)|$ is associated with a wind flow undergoing continuous deceleration and thus the Poynting flux increases with distance. Therefore we conclude on these physical grounds that the circumpolar current cannot remain finite at infinity.

4.3. Asymptotic Proper Current for Relativistic Winds.

For relativistic winds, an entirely similar analysis (Heyvaerts and Norman 2002b) has shown that relations similar to Eq.(41) and (42) hold true, namely

$$\frac{\Gamma}{\Gamma-1} Q_0 \rho_0^{\Gamma-1}(b) = \frac{\Omega_0 K(b)}{\alpha_0} \tag{46}$$

$$\frac{\lambda_r(n_0(b))}{n_0^{\Gamma-1}(b)} = (2-\Gamma)\ \ln(n_0(b))\ +\ \ln\left(\frac{4b^2}{\ell^2}\right) \tag{47}$$

where λ_r, defined by

$$\lambda_r = \int_0^A \frac{da'}{a_0} \frac{\Omega(a')\sqrt{c^2 + K(b)\Omega(a')/\alpha(a')}}{\sqrt{E^2(a') - (c^2 + K(b)\Omega(a')/\alpha(a'))^2}}\ \frac{\sqrt{E_0^2 - (c^2 + K(b)\Omega_0/\alpha_0)^2}}{\Omega_0\sqrt{c^2 + K(b)\Omega_0/\alpha_0}} \tag{48}$$

is a function of $n_0(b)$, since $K(b)$ is given by Eq.(46). Similar arguments then show that, if K is to approach a finite limit, its value should be the absolute minimum of the function $(\alpha(E-c^2)/\Omega)$, reached at a non-zero regime change flux, a_*. So, if K_∞ is not zero:

$$|K_\infty| = K_{sup} = \mathrm{Min}_{a\neq 0}\ |\frac{\alpha(a)(E(a)-c^2)}{\Omega(a)}| \tag{49}$$

This limit, however, can only be approached from below, again leading to a physical contradiction. Although K is not directly related to the poloidal current I anymore, it is possible to see that, if the wind source is the only current source, K should be a decreasing function of b. Indeed, the asymptotic limit of K, $K_\infty(b)$, is related to the asymptotic current $I_\infty(a,b)$ enclosed in surface a at distance b by (Heyvaerts and Norman 2002b)

$$I_\infty(a,b) = \gamma_\infty(a,b) K_\infty(b) \tag{50}$$

where $\gamma_\infty(a,b)$ is the Lorentz factor on surface a at distance b. Again, $I_\infty(a,b)$ decreasing with b, at constant a, implies plasma acceleration, since the field-aligned poloidal Lorentz force $\vec{j}_P \times \vec{B}_\theta$ is then in the sense of the motion. This causes $\gamma_\infty(a,b)$ to increase (Begelman and Li 1994). Thus, $I_\infty(a,b)$ decreasing with b implies that $K_\infty(b)$ is also decreasing with b. Solutions with a finite asymptotic value of $|K|$, which can be approached only by increasing values of $|K|$, are therefore physically inconsistent. The circumpolar proper current cannot be finite at infinity for relativistic winds.

4.4. Asymptotic Current: Progressive Deconfinement

That the circumpolar current can only be zero or given by Eq.(45) can also be understood by the following analysis. Let us suppose that an external pressure independent of z confines

the jet in such a way that it has cylindrical geometry. At low confining pressure, its structure, which is entirely determined by the first integrals and by the boundary condition, will consist of a circumpolar current-carrying channel surrounded by a large, almost pressureless, field region carrying very little current density. In this latter region, the current enclosed in a magnetic surface a is almost independent of a and equal to I. This field-region value of I depends on the external confining pressure P_{ext}. We calculate explicitly in Appendix (A) the relation between I and P_{ext}, both when the edge of the jet is either a magnetic surface or a null surface. The conclusion of this analysis is that I approaches either zero or I_{sup} (K_{sup} in the relativistic case) as P_{ext} decreases to zero.

4.5. Asymptotic Current: Conclusion

The discussions in subsections (4.2) and (4.3) have clearly shown that if the wind extends over very large distances, the amount of circumpolar electric current should approach zero as z approaches infinity. When this terminal regime is reached, all of the wind energy is in kinetic form. As shown in Heyvaerts and Norman (1989), Heyvaerts and Norman (2002a) and Heyvaerts and Norman (2002b), the magnetic surfaces are then paraboloidal.

5. Characteristic MHD Speeds

5.1. Terminal Speed *vs* Fast Mode Speed

In the absence of external pressure, a wind should eventually achieve, on each field line, a terminal velocity exceeding all the MHD characteristic velocities. We now show that the fully asymptotic wind regime cannot satisfy this condition if the wind is to carry a finite circumpolar current. Specifically, super fast mode velocities cannot be reached in the vicinity of the regime-change surface a_* in the field-region. As discussed above, the terminal velocity has to vanish at this surface and, for a classical wind, the circumpolar current should equal I_{sup}, the minimum value of the function $(\alpha E/\Omega)$. Does the terminal velocity, though vanishing, remains larger than the fast mode speed? A necessary, though not sufficient, condition for the flow remaining super-fast-mode is that the fast mode velocity itself approaches zero as a approaches a_*.

The position and velocity at the fast critical point are obtained from the Bernoulli equation (5). Let us write it, for a given magnetic surface a, in the form $\mathcal{B}(r, \rho) = 0$. The

critical points and associated velocities are obtained by solving the system

$$r\frac{\partial \mathcal{B}}{\partial r} = -\frac{1}{2}\,\frac{\alpha^2}{\rho^2}\, r\, \frac{dB_P^2}{dr} - r\frac{d\Phi_G}{dr} + \Omega^2 r^2 + \frac{\rho_A^2}{r^2}\frac{L^2 - \Omega^2 r^4}{(\rho_A - \rho)^2} = 0 \tag{51}$$

$$-\rho\frac{\partial \mathcal{B}}{\partial \rho} = -\frac{\alpha^2 B_P^2}{\rho^2} + \Gamma Q \rho^{\Gamma-1} + \frac{\rho\rho_A^2}{r^2}\,\frac{(L-\Omega r^2)^2}{(\rho_A - \rho)^3} = 0 \tag{52}$$

The velocity at the fast point is then, for classical winds :

$$v_f^2 = \Gamma Q \rho^{\Gamma-1} + \Omega^2 \frac{\rho\rho_A^2 (r^2 - r_A^2)^2}{r^2(\rho_A - \rho)^3} \tag{53}$$

A lower bound to this is

$$v_f^2 \geq \Omega^2 \frac{\rho\rho_A^2 (r^2 - r_A^2)^2}{r^2(\rho_A - \rho)^3} \tag{54}$$

which, from Eqs. (9) and (2) means that $v_f^2 \geq (B_P^2 + B_\theta^2)/\mu_0\rho$. A weaker, but accurate, lower bound for v_f^2 is $B_\theta^2/\mu_0\rho$, which is obtained in the limit $r \gg r_A$ and $\rho \ll \rho_A$. Using Eq.(24), the inequality (54) reduces, in this limit, to

$$v_f^2 \geq |\frac{I\Omega}{\alpha}| \tag{55}$$

The fluid terminal velocity $v_\infty(a)$, given by Eq.(28), cannot remain larger than $I\Omega/\alpha$ if v_∞ is to approach zero as a approaches a_*, since $|I|$ is supposedly close to a non-zero value and α supposedly remains finite. Note however that this would not be so if a_* were a neutral magnetic surface, where I vanishes and α diverges. A more general argument is as follows. Eq.(28) and the inequality (55) indicate that the wind can only be super fast mode if :

$$|I| \leq \frac{2}{3}\;\frac{\alpha E}{\Omega} \tag{56}$$

In the field-regions of the asymptotic domain, $|I|$ is a constant, I_∞. $|I|$ should be less than I_{sup}, the minimum value of $|\alpha E/\Omega|$, otherwise, from Eq.(28), v_∞^2 would not be positive everywhere. If it is required that the wind be everywhere super-fast mode at large distances, inequality (56) shows that the upper limit I_{sup} cannot be reached. In particular, the cylindrical asymptotics regime,for which $I_\infty = I_{sup}$, is not consistent with the wind being everywhere super-fast-mode. Winds in an intermediate regime, which deliver a circumpolar current very close to the maximum I_{sup}, should still be in the sub-fast mode regime. Unless the environment of the jet makes it possible that its velocity does not exceed the fast mode speed asymptotically, the electric current should eventually fall below $\frac{2}{3}I_{sup}$ at very large distances. This argument again points to the conclusion that the completely asymptotic regime should have a vanishingly small circumpolar electric current.

5.2. Fast Point in the Relativistic Regime

Similar conclusions are found in the case of relativistic winds. The location and velocity of the fast point can be found as in the classical case. Neglecting gravity and the gas entropy, the relativistic Bernoulli equation takes the form

$$\left(c^2 + \frac{\rho r^2 \Omega^2}{\mu_0 \alpha^2}\right)^2 \left(1 + \frac{u_p^2}{c^2}\right) = E^2 \tag{57}$$

Assuming the shape of field lines is given, Eq.(14) turns Eq.(57) into an equation for u_p, $\mathcal{B}(u_p, r) = 0$. At critical points the differential of $\mathcal{B}$ vanishes. The position, r_f, of the fast point is then given by :

$$\frac{\partial(B_p r^2)}{\partial s} = 0 \tag{58}$$

where s is the curvilinear abcissa along a field line. Eq.(58) can be satisfied at large distances in any particular geometry compatible with the Bernoulli equation (appendix (B)). For an initial monopolar field at the source of a highly magnetized, relativistic wind the fast point is at a large distance from the light cylinder (Beskin et al. 1998). The specific momentum at the fast point is obtained from the condition that $\partial\mathcal{B}/\partial u_p$ vanishes. The Lorentz factor associated with the poloidal velocity at the fast point is:

$$\gamma_{pf} = \left(\frac{E}{c^2}\right)^{\frac{1}{3}} \tag{59}$$

Eq.(57) can be solved when the flow is in the asymptotic domain, since ρr^2 becomes a constant at large distances, resulting in Eq.(36) which relates the asymptotic specific momentum to the proper total current K. A condition for the flow to be super-fast-mode, similar to the inequality (56), can be derived for relativistic flows. The fast mode speed, c_f, for a diffuse highly magnetized medium where the Alfvén speed $c_A = (B^2/\mu_0\rho)^{1/2})$ may exceed the speed of light, is given by

$$\frac{1}{c_f^2} = \frac{1}{c^2} + \frac{1}{c_A^2} \tag{60}$$

The Alfvén speed associated with the total field, c_A, is larger than the the Alfvén speed associated with the toroidal field, $c_{A\theta}$. This gives for c_f the following inequality:

$$c_f^2 \geq \frac{c_{A\theta}^2 c^2}{c^2 + c_{A\theta}^2} \tag{61}$$

The proper current K is related to I and γ by Eq.(50) and can be expressed in terms of ρ by (Heyvaerts and Norman 2002b):

$$K = \frac{\rho r^2 \Omega}{\mu_0 \alpha} \tag{62}$$

Using these relations, the inequality (61) can be transformed into the following inequality for the specific momentum $u_f = c_f \gamma(c_f)$ associated with c_f:

$$u_f^2 \geq E^2 \frac{K\Omega/\alpha}{\left(c^2 + K\Omega/\alpha\right)^2} \tag{63}$$

A necessary condition for the relativistic flow to be super-fast-mode is that the terminal specific momentum of the wind flow, $u_{P\infty}$, given by Eq.(36), exceeds the lower bound to the asymptotic value of u_f, given by Eq.(63). This can be written as:

$$c^2 \left(E^2 - \left(c^2 + \frac{K\Omega}{\alpha} \right)^2 \right) \geq E^2 \frac{K\Omega}{\alpha} \tag{64}$$

Now, if the asymptotic flow regime is to have a finite value of the circumpolar proper current K, the latter must be the minimum value of $\alpha(E - c^2)/\Omega$, reached at some non-zero a_*. At this particular value of a, the inequality (64) cannot be satisfied, since its left side vanishes whereas its right side is strictly positive. Expanding (64) for E close to c^2 gives again the inequality (56) for classical dynamics. Therefore, relativistic winds in an intermediate regime having a circumpolar current very close to the maximum I_{sup} should still be in the sub-fast mode regime in some regions. Since the fast mode critical surface is unlikely to be found far in the asymptotic domain, this situation should be exceptional. By contrast, the completely asymptotic relativistic regime should have a vanishingly small circumpolar electric current.

6. Intermediate Asymptotic Regime

6.1. Is the Mathematical Asymptotic Regime ever Reached?

In the fully asymptotic regime, the currents I and K should approach zero with increasing distance b to the wind source. However, our calculations have shown that they do so only very slowly, decreasing as the inverse of the logarithm of the distance to the source. This opens the question of whether, in fact, when the flow reaches the terminal shock separating it from the outer medium, I has indeed vanished.

If not, the region in the wind cavity at $r \gg r_A$ would be in an intermediate asymptotic regime, i.e., one in which I is still finite, though smaller than I_{sup}. For classical winds, the flux distribution would then be as given by Eq.(32), with the value of $\rho_0(b)$ and $I(b)$ in the asymptotic domain given by Eqs.(41) and (42). Extreme focusing would be possible for I close to its maximum value, I_{sup}, defined in Eq.(45). However such a wind would not yet have made its transition through the fast critical point. Similar considerations apply to relativistic winds.

Such an intermediate asymptotic state is associated with a finite value of $I < I_{sup}$ (or $K < K_{sup}$). It is appropriate to discuss the properties of winds which carry a current close to the maximum possible value, since these have strong focusing properties and may be directly related to ubiquitous jet phenomena. We show below that such winds have very special properties. For example, when the current is close to I_{sup}, the wind splits into a focused jet and an equatorial wind with radial geometry. In between these two components, there is only a very weak flow, but a large Poynting flux. Such highly focused winds must be sub fast mode as we show below.

6.2. Poynting-Flux Dominated Winds: Polar Jets and Equatorial Winds

We now discuss the structure of rotating MHD winds which carry a circumpolar current close to the maximum possible value. This current is treated as remaining constant in the intermediate region as discussed above. The wind's structure is given by Eq.(32). Suppose that I is close to its maximum possible value, I_{sup}, the minimum of the function $(\alpha E/\Omega)$, reached at some $a_* \neq 0$. It is assumed that $(\alpha E/\Omega)$ has such a minimum. In this case, the integral in Eq.(32) would almost diverge at a_*. As a result, all the flux at $a \leq a_*$ is strongly focused about the axis. The focusing is still conical however because divergence is not exactly reached. The flux at $a \geq a_*$ is spread about the equator. The terminal flow velocity in the vicinity of $a = a_*$ almost vanishes, while the angular extent of this region is large. This structure is represented in Fig.(2).

A wind close to the maximum current then appears to have a rather peculiar structure. It is split into two main flow regions, a circumpolar flow channeled in a cone of very small opening angle, and a conically spread equatorial flow. These two regions are separated by a region with very little flow, where the outflowing energy is almost entirely in Poynting form. This result has been also obtained in numerical simulations by Ustyugova et al. (2000) of the flow from a Keplerian disk threaded by a dipole-like magnetic field.

Again, from the discussion of section (4), this regime cannot be the eventual asymptotic regime. The latter necessarily involves a vanishing circumpolar current. The solutions described above occur in the intermediate asymptotic regime , which may persist over large distances. That I or K are close to their maximum possible value, I_{sup} or K_{sup}, has however here the character of an extra and rather conjectural assumption.

In the particular case when the function $(\alpha E/\Omega)$ reaches its minimum at $a = 0$, the situation regarding the focusing of the flow is different. The flux function a is proportional to r^2 near the polar axis. Symmetry about the axis implies that any physical quantity be even

in r, and reaches a flat extremum at $r = 0$. This however does not imply that functions of the flux a should behave similarly. On the contrary, a linear variation with a should be the rule. Therefore when $a_* = 0$ and the circumpolar current is supposedly close to $\alpha_0 E_0/\Omega_0$, there is no divergence of $\tan\psi(a)$ (Eq.(32)), as a approaches zero, due to the vanishing of the square root in the denominator, because the latter usually has a simple zero. Most of the mass flux is then spread about the equator in a wind shaped like a conical fan. This structure is represented in Fig.(3). There is little flow about the axis, since the velocity, equal to $\sqrt{2(E - I\Omega/\alpha)}$, is small in this region. Eq.(32) shows that the wind nevertheless fills all space. Indeed, $\psi(a)$ approaches $\pi/2$ as a approaches zero because I vanishes on the axis. The flow near the axis would in this case be weak, the energy being transported mainly in electromagnetic form in a circumpolar flux tube subtending negligible flux.

6.3. Neutral Magnetic Surfaces do not Focus Winds

In section (6.2) above, a wind carrying nearly maximal current in an intermediate asymptotic regime has been considered. This situation induces a strong focusing of magnetic surfaces. Note that at a neutral surface, where I also vanishes, Eq.(32) does not give rise to a divergence of $\tan\psi(a)$, since the integral in Eq.(32) is convergent (Heyvaerts and Norman 2002a,b). Neutral surfaces do not force the magnetic surfaces nested in them to cylindrical shapes. It is shown in Appendix(A) that a neutral magnetic surface at the jet's edge does not cause the radius of a pressure-confined jet to diverge. We conclude that currents returning at neutral magnetic surfaces do not cause any general focusing of the structure. Okamoto (1999) shows that they rather have a defocusing effect.

7. Conclusions on the Asymptotic Regime of MHD Winds

The major conclusions of this paper are given below.

1. If there were to be a finite circumpolar current at infinity then, for classical winds, this current should have a value equal to the minimum value of the function $\alpha E/\Omega$. This minimum value, I_{sup}, is the maximum allowed current. If this maximum current is achieved,then there exists a particular magnetic surface, a_*, where $\alpha E/\Omega$ reaches its minimum and where the total energy is carried by Poynting flux alone.

2. The limiting maximum current, I_{sup}, can, however, only be reached from below. In other words, as the distance from the source increases, the circumpolar current would

have to increase. This cannot be achieved fro winds emanating from a finite, central source of magnetic flux. Thus, we conclude that, for non-pathological cases, the maximal current cannot be reached from below. Therefore, the asymptotic circumpolar current must vanish.

3. A winds carrying a cicumpolarcurrent, I, that is bounded from below such that, $I > \frac{2}{3}I_{sup}$, cannot have a terminal velocity everywhere in excess of the fast mode speed.

4. The circumpolar current declines to zero only inversely logarithmically with distance. An important consequence of this very slow decline is that there exists an extensive intermediate asymptotic regime, where this current is still finite. Thus, significant Poynting flux can be carried over very large distances. The termination of the wind, for example at a terminal shock, may occur before this current has decayed to zero and the σ parameter is still large

Similar conclusions apply to relativistic winds:

1. The total circum-polar proper current K is, for relativistic winds, less than K_{sup}, the minimum value of the function $\alpha(E - c^2)/\Omega$. Any finite value of this proper current at infinity equals K_{sup}. If this maximum current were to be achieved asymptotically, there would exist a particular magnetic surface, a_*, where $\alpha(E - c^2)/\Omega$ reaches its minimum value and where the total energy is carried by Poynting flux alone.

2. The upper bound to the proper current, K_{sup}, can only be reached from below. As in the non-relativistic case, it is not physically possible to reach this upper bound for winds emanating from a finite, central source of magnetic flux. Thus, the circum-polar proper current should formally approach zero at infinity.

3. Winds that carry a proper current such that inequality (64) is not satisfied cannot have terminal velocities everywhere in excess of the fast mode speed.

4. The circumpolar total proper current declines to zero only inversely logarithmically with distance. A consequence of this very slow decline is that there is an extensive intermediate asymptotic regime where this current is still finite. Significant Poynting flux can be carried over large distances, and the wind may terminate in a shock much before this current has decayed to zero.

5. In the intermediate asymptotic regime, a classical or relativistic wind carrying a proper current close to the maximum allowed value I_{sup} (or K_{sup}) consists of a conical wind of very small opening angle about the pole surrounded by a region in which most of the energy flux is in Poynting form.

6. Such close-to-maximum Poynting Flux carrying winds with polar jets and equatorial winds must be sub-fast mode. External boundary conditions may significantly influence their nature.

7. For force-free initial conditions in the near field, the energy close to the source is all Poynting flux. Thus, relativistic jets emanating from a strongly magnetized base can maximize the output in Poynting flux

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A. Unconfined Jets as limits of cylindrical pressure-confined jets

We analyze the structure of a pressure-confined cylindrical jet with a given set of first-integrals as the confining pressure is reduced. For completeness, we consider both the case of a jet bordered by an ordinary magnetic surface and the case of a jet enclosed by a neutral magnetic surface. This is appropriate since the electric current organizes itself into closed cells and because the presence of a neutral magnetic surface at the jet's edge slightly changes its structure, favoring space filling.

Where a uniform external pressure imposes cylindrical geometry for the magnetic surfaces, the orthogonal trajectories are straight lines perpendicular to the rotation axis. Taking into account the smooth matching with the polar boundary layer the radius $r(a)$ of the cylindrical flux surface a is found to be:

$$r^2(a) = \frac{\Gamma}{\Gamma-1}\, \frac{Q_0 \mu_0 \alpha_0^2 \rho_0^{\Gamma-2}}{\Omega_0^2} \exp\left(\int_0^a \frac{\sqrt{2}\Omega(a')da'}{\mu_0 I \sqrt{E(a') - I\Omega(a')/\alpha(a')}}\right) \tag{A-1}$$

The variables I and ρ_0 are related by the Bennet relation

$$\frac{\Gamma}{\Gamma-1}\, Q_0 \rho_0^{\Gamma-1} = \frac{I\Omega_0}{\alpha_0} \tag{A-2}$$

They both depend on the external confining pressure, P_{ext}, as does also the radius r_* of the outermost magnetic surface. The boundary condition imposes equilibrium between the externally applied pressure and the total pressure at the edge of the jet. The latter is the sum of toroidal magnetic pressure and gas pressure since poloidal magnetic pressure can be neglected. If a_* is simply an ordinary magnetic surface, gas pressure can also be neglected and the boundary condition becomes:

$$P_{ext} = \frac{\mu_0 I^2}{2\, r_*^2} \tag{A-3}$$

where r_* depends on I as given by Eq.(A-1). In this case, I and P_{ext} are simply related by

$$2\left(\frac{\Gamma}{\Gamma-1}\right)^{\frac{1}{\Gamma-1}} \left(\frac{\alpha_0 Q_0 \rho_{A0}^{\Gamma-1}}{I\Omega_0}\right)^{\frac{\Gamma}{\Gamma-1}} \exp\left(\int_0^{a*} \frac{\Omega(a')da'}{\sqrt{2}\mu_0 I \sqrt{E(a') - I\Omega(a')/\alpha(a')}}\right) = \frac{Q_0 \rho_{A0}^{\Gamma}}{P_{ext}} \tag{A-4}$$

As P_{ext} decreases, the left hand side of Eq.(A-4) should diverge, which leaves at most two solutions for I, $I = 0$ and, if $(\alpha E/\Omega)$ has a flat minimum, $I = I_{sup}$ (Heyvaerts and Norman (2002a), Heyvaerts and Norman (2002b), Lery et al. (1999)). The value I_{sup} is reached by increasing I as P_{ext} decreases.

Some complications arise when the surface a_* is a neutral one. The jet is then bordered by a neutral surface. The proof that $I = 0$ or I_{sup} in this unconfined limit should be extended to this case. The outskirts of the pressure-confined jet then consist of half a sheet pinch boundary layer, in which the transfield equilibrium reduces approximately, $\vec{n}$ being the unit vector normal to magnetic surfaces, to:

$$\vec{n} \cdot \vec{\nabla} \left(Q\rho^\Gamma + \frac{B_\theta^2}{2\mu_0} \right) = 0 \tag{A-5}$$

B_θ is given by Eq.(24). Integrating accross the boundary layer, we get, taking into account the divergence of the function α at the null surface:

$$Q\rho^\Gamma + \frac{1}{2}\frac{\rho^2 r^2 \Omega}{\mu_0 \alpha^2} = Q_* \rho_*^\Gamma \tag{A-6}$$

where ρ_* is the density at the jet's edge. It does not vanish, because of the confining pressure. Treating Q, E and Ω as constants equal to Q_*, E_* and Ω_* in this external boundary layer, Eq.(A-6) provides an expression for α as a function of the parameter

$$X = \frac{\rho}{\rho_*(b)} \tag{A-7}$$

namely:

$$\frac{1}{\mu_0 \alpha^2} = \frac{2 Q_* \rho_*^{\Gamma-2}}{\Omega_*^2 r^2} \left(\frac{1}{X^2} - \frac{1}{X^{2-\Gamma}} \right) \tag{A-8}$$

Eq.(A-8) implicitly gives a as a function of X for known $\alpha(a)$. The function α cannot be treated as approximately constant because of its divergence at $a = a_*$. Nevertheless, in our analysis, α is a given function of $(a_* - a)$ in this vicinity. It is expected that r itself should vary only little in the boundary layer if the latter is indeed thin, which will be checked *a posteriori*. The flux distribution versus radius results from Eq.(2), the velocity being obtained from Eq.(5). Disregarding the gravitational potential and the toroidal velocity and assuming that the velocity at the jet's edge is largely supersonic, we obtain

$$r\frac{dr}{da} = \frac{\alpha(a)}{\rho_* \; X \; \sqrt{2E_*}} \tag{A-9}$$

Eq.(A-9) implicitly gives $r(X)$ and Eq.(A-8) implicitly gives $a(X)$. To derive an explicit parametric representation of the flux radius relation in the boundary layer we deduce $a(X)$ from Eq.(A-8). We have shown (Heyvaerts and Norman 2002a) that in the vicinity of the neutral surface $\alpha(a)$ varies as:

$$\frac{1}{\alpha(a)} = \frac{(a_* - a)^\nu}{\eta a_*^\nu} \tag{A-10}$$

where η is a constant having the same dimension as α and ν is a positive exponent, strictly smaller than unity and most often equal to (1/2). This implies that the edge boundary layer is not infinitely extended. Indeed, as a approaches a_*, X becomes close to unity and, in this limit, the relation between radius and α as given by Eqs.(A-9) and (A-10) reduces to:

$$r\ dr \sim \alpha^{-1/\nu}\ d\alpha \tag{A-11}$$

Since $(1/\nu)$ is strictly larger than unity, this implies that, as α diverges near a_*, r remains bounded. This result is similar to our finding (Heyvaerts and Norman 2002a) that the angular thickness of a neutral sheet boundary layer remains limited and in fact small. Assume, similarly, that the boundary layer thickness remains small in this case, so that r varies only little. The boundary layer thickness is then obtained, using Eqs.(A-8) and (A-10), by turning Eq.(A-9) into the following differential equation for $r(X)$:

$$2r\frac{dr}{dX} = \frac{\eta a_*}{\rho_*\sqrt{2E_*}}\left(\frac{2Q_*\mu_0\eta^2\rho_*^{\Gamma-2}}{\Omega_* r_*^2}\right)^{\frac{1}{2\nu}-\frac{1}{2}}\ \left(\frac{2-(2-\Gamma)X^\Gamma}{X^{\frac{1}{\nu}}(1-X^\Gamma)^{\frac{3}{2}-\frac{1}{2\nu}}}\right) \tag{A-12}$$

Integrating, the thickness Δ_* of the edge boundary layer is found. Let us denote by $k(\nu,\Gamma)$ the integral

$$k(\nu,\Gamma) = \int_{\frac{1}{2}}^{1}\frac{\left(2-(2-\Gamma)X^\Gamma\right)\ dX}{X^{\frac{1}{\nu}}(1-X^\Gamma)^{\frac{3}{2}-\frac{1}{2\nu}}} \tag{A-13}$$

which is convergent at $X=1$ since ν is less than unity. From Eq.(A-12) we get:

$$2r_*\Delta_* = \frac{k(\nu,\Gamma)\eta a_*}{\rho_*\sqrt{2E_*}}\left(\frac{2Q_*\mu_0\eta^2\rho_*^{\Gamma-2}}{\Omega_* r_*^2}\right)^{\frac{1}{2\nu}-\frac{1}{2}} \tag{A-14}$$

The total outer radius r_* of the jet, when under the pressure P_{ext}, is obtained by adding Δ_* to the extent of the domain occupied by the flux tube at the inner limit of the edge boundary layer. Since this flux is close to being a_*, this radius is almost given by Eq.(A-1) with $a=a_*$. The total poloidal current is treated as if it were constant up to a_*, whereas in fact it decreases to zero in the boundary layer. Then we find:

$$\begin{aligned} r_*^2 = \frac{\Gamma}{\Gamma-1}\ \frac{Q_0\mu_0\alpha_0^2\rho_0^{\Gamma-2}}{\Omega_0^2}\exp\left(\int_0^{a_*}\frac{\sqrt{2}\Omega(a')da'}{\mu_0 I\sqrt{E(a')-I\Omega(a')/\alpha(a')}}\right) \\ +\frac{k(\nu,\Gamma)\eta a_*}{\rho_*\sqrt{2E_*}}\left(\frac{2Q_*\mu_0\eta^2\rho_*^{\Gamma-2}}{\Omega_* r_*^2}\right)^{\frac{1}{2\nu}-\frac{1}{2}} \end{aligned} \tag{A-15}$$

The edge density ρ_* is related to P_{ext} by the boundary condition

$$P_{ext} = Q_*\rho_*^\Gamma \tag{A-16}$$

The neutral layer radius is related to P_{ext} and to I by the neutral surface's Bennet relation,

$$\mu_0 I^2 = 2Q_* \rho_*^\Gamma r_*^2 \tag{A-17}$$

The axial density ρ_0 is related to I by Eq.(A-2). Eliminating ρ_*, ρ_0 and r_* in favour of I, Eq.(A-15) takes the form of a relation between I and P_{ext}, which is best written by introducing the dimensionless variables D and i defined by:

$$\rho_* = \mu_0 \alpha_0^2 D \tag{A-18}$$

$$I = \frac{\alpha_0 E_0}{\Omega_0} i \tag{A-19}$$

The relation between I and P_{ext} then translates into the following relation between i and D:

$$\frac{2Q_*}{Q_0}\left(\frac{\Gamma}{\Gamma-1}\right)^{\frac{1}{\Gamma-1}}\left(\frac{Q_0\rho_{A0}^{\Gamma-1}}{i\ E_0}\right)^{\frac{\Gamma}{\Gamma-1}} \exp\left(\frac{\sqrt{2}\Omega_0^2 a_*}{\mu_0\alpha_0 E_0^{3/2}}\int_0^{a_*}\frac{\Omega(a')da'}{i\ \Omega_0 a_*\sqrt{\frac{E(a')}{E_0} - i\ \frac{\Omega(a')\alpha_0}{\Omega_0\alpha(a')}}}\right)$$
$$+\frac{2^{\frac{2-\nu}{\nu}}k(\nu,\Gamma)a_*\Omega_0^2}{\mu_0\alpha_0^2\ E_0^{3/2}}\left(\frac{E_0}{E_*}\right)^{\frac{1}{2}}\left(\frac{\eta}{\alpha_0}\right)^{\frac{1}{\nu}}\left(\frac{\Omega_0}{\Omega_*}\right)^{\frac{1}{\nu}-1}\left(\frac{Q_*\rho_{A0}^{\Gamma-1}}{E_0}\right)^{\frac{1}{\nu}}\frac{D^{\frac{\Gamma-1}{\nu}-\Gamma}}{i^{\frac{1}{\nu}+1}} = \frac{1}{D^\Gamma} \tag{A-20}$$

which is of the form:

$$\Lambda_0\ \frac{\Phi(i)}{i^{\Gamma/(\Gamma-1)}} + \Lambda_1 \frac{D^{\frac{\Gamma-1}{\nu}-\Gamma}}{i^{\frac{1}{\nu}+1}} = \frac{1}{D^\Gamma} \tag{A-21}$$

The coefficients Λ_0 and Λ_1 are positive and $\Phi(i)$ is:

$$\Phi(i) = \exp\left(\frac{\sqrt{2}\Omega_0^2 a_*}{\mu_0\alpha_0 E_0^{3/2}}\int_0^{a_*}\frac{\Omega(a')da'}{i\ \Omega_0 a_*\sqrt{\frac{E(a')}{E_0} - i\ \frac{\Omega(a')\alpha_0}{\Omega_0\alpha(a')}}}\right) \tag{A-22}$$

When the external pressure becomes very small, the second term on the left-hand side of Eq.(A-21) becomes negligible if i remains finite and a non-vanishing solution is then found for $i = i_{sup}$, the value of i that causes the function $\Phi(i)$ to diverge. If $\Phi(i)$ does not diverge for finite i, the solution for i as D becomes small moves towards $i = 0$. This solution exists also when $\Phi(i)$ diverges for finite i. The second term on the left hand side of Eq.(A-21) is then again negligible with respect to the first one because of the very rapid divergence of $\Phi(i)$ as i approaches zero. The solution is then close to that of the equation

$$\frac{1}{D^\Gamma} = \Lambda_0 \frac{\Phi(i)}{i^{\Gamma/(\Gamma-1)}} \tag{A-23}$$

We conclude that in this limit of vanishing external pressure the outer boundary layer has negligible effect on the relation between the current and the external pressure. Actually, Eq.(A-23) is just the non-dimensional form of Eq.(A-1). As a result $I = 0$ and $I = I_{sup}$ remain the only possible solutions in the limit of deconfinement, even when the jet is bordered by a neutral magnetic surface.

B. Fast Critical Point Position

For conical asymptotics the function $|r\nabla a|$ is bounded from above and below with $\mu(a) \leq |r\nabla a| \leq \lambda(a)$ as (r/r_A) tends to infinity (Heyvaerts and Norman 1989). Thus, for conical asymptotics $(d(r|\nabla a|)/ds)$ approaches 0. We assume that there is only one fast point in this analysis. For cylindrical asymptotics, $(d|r\nabla a|/ds)$ also approaches 0 and the fast point may be at infinity. For parabolic field lines, representable as by $z = K(a)r^{p(a)}$, say, $r|\nabla a|$ is given by:

$$r|\nabla a| = \frac{\sqrt{p^2(a) + r^2/z^2}}{|K'/K + p' log r|} \tag{B-1}$$

For non-constant $p(a)$, $r|\nabla a|$ approaches 0, as r approaches infinity. If $p(a)$ is a constant larger than unity, $r|\nabla a|$ approaches a constant value and $d(r|\nabla a|)/ds$ approaches zero. If $p(a) = 1$, r/z goes to a constant as s approaches infinity, and again $d(r|\nabla a|)/ds$ approaches 0. In general, $d(r|\nabla a|)/ds$ becomes very small at large distances, allowing a fast critical point to be located in the asymptotic region.

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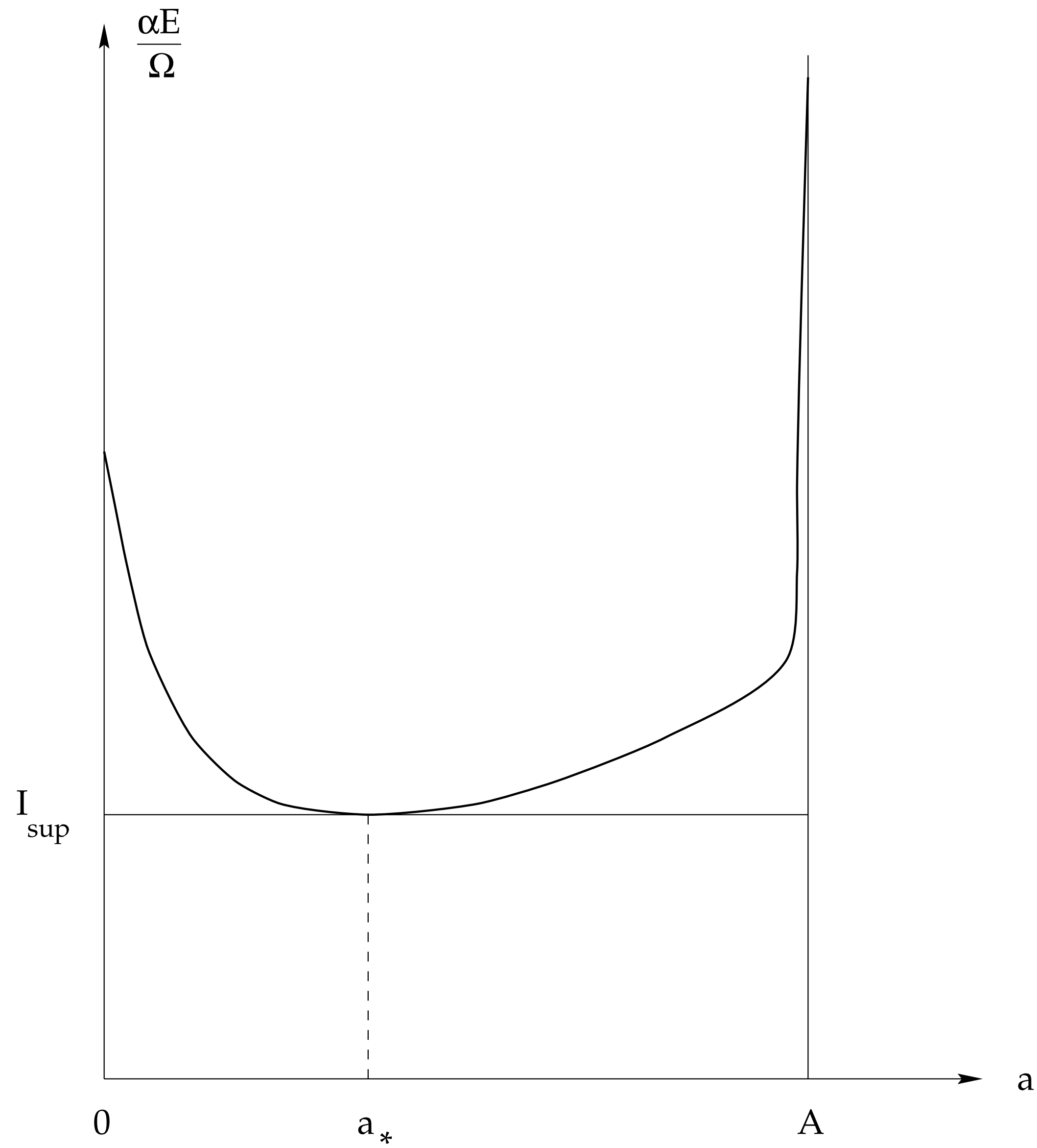

Fig. 1.— The function $(\alpha E/\Omega)$ versus flux a for some classical wind and for a poloidal field of supposedly dipolar symmetry. The equatorial plane, at $a = A$, is the only null surface. Both α and $(\alpha E/\Omega)$ diverge at a null surface. The current I (Eq.(23)) can only take values between zero and I_{sup}, the minimum value of $(\alpha E/\Omega)$. This function is assumed here to have an absolute minimum at some a_* away from the polar axis. The value of I at infinitely large distances could then be zero or I_{sup} (see however sections (4.2) and (4.3)). The flux contained in a cylindrical jet about the axis at such large distances must be $2\pi a_*$ (section 4.1). For some winds, the function $\alpha E/\Omega$ could be monotonically increasing. In this case I could only vanish at infinitely large distances. For relativistic winds, the function $(\alpha(E - c^2)/\Omega)$ should be substituted for $(\alpha E/\Omega)$ and the proper current K (Eq.(50)) for I, with no other change in the above analysis

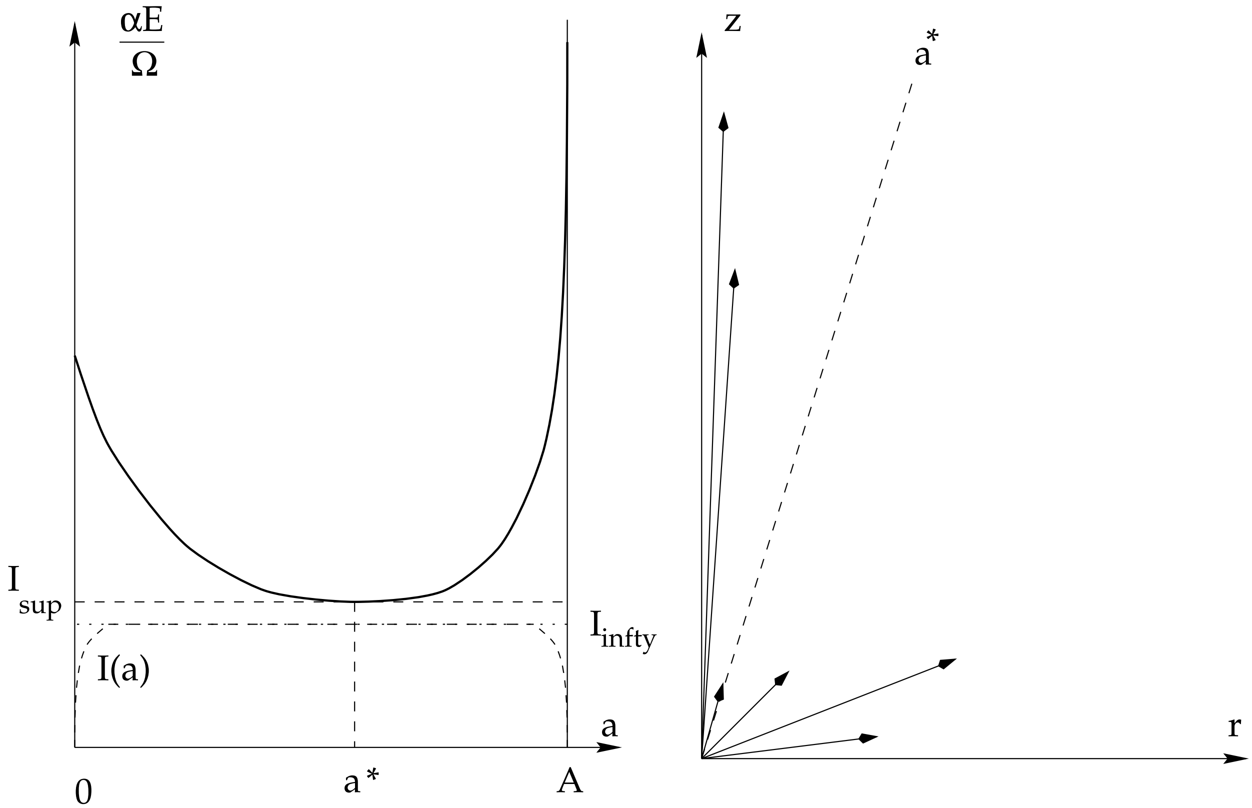

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Fig. 2.— A near-maximal current jet. In an intermediate asymptotic regime, the circum-polar current, I_∞, may still be close to the maximal value I_{sup}. The left panel represents $(\alpha E/\Omega)$ (or $(\alpha(E-c^2)/\Omega)$ for a relativistic wind), I_{sup} and I_∞ (or K_{sup} and K_∞ resp.), as well as the run with a of the enclosed total current $I(a)$ (resp. $K(a)$), taking into account the polar ($a=0$) and equatorial ($a=A$) current-carrying boundary layers. The wind structure (Eq.(33) is represented in the right panel. Each arrow indicates the wind speed and direction for a set of equidistant values of the flux variable a. The dashed line represents the magnetic surface $a=a_*$. The regions close to the pole and the equator are boundary layers where the circum-polar current and return current flow. These bounday layers are not explicitly represented.

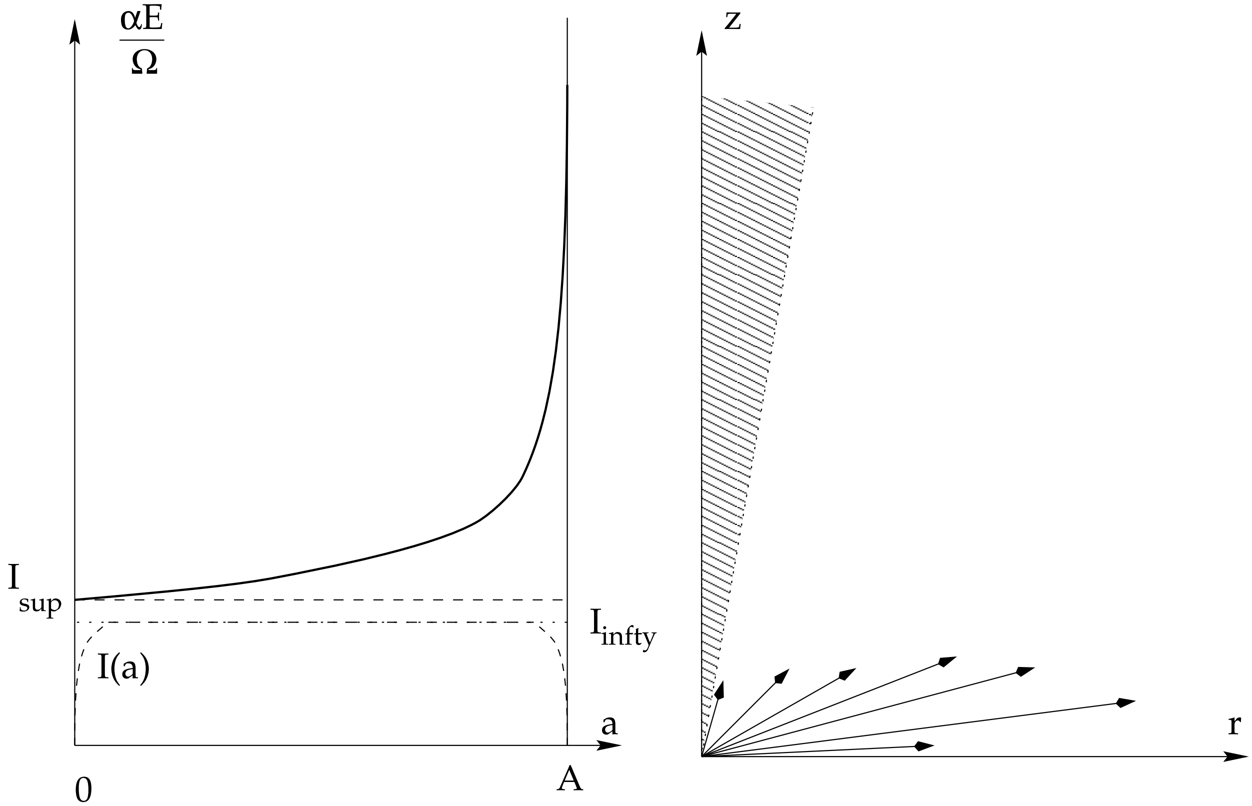

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Fig. 3.— A near-maximal current wind. In an intermediate asymptotic regime,the circumpolar current, I_∞, may still be close to the maximal value I_{sup}. In the present case, this maximum value is supposedly reached at the polar axis, $a = 0$. The left panel represents $(\alpha E/\Omega)$ (or $(\alpha(E - c^2)/\Omega)$ for a relativistic wind), I_{sup} and I_∞ (or K_{sup} and K_∞ resp.), as well as the run with a of the enclosed total current $I(a)$ (resp. $K(a)$), taking into account the polar ($a = 0$) and equatorial ($a = A$) current-carrying boundary layers. The wind structure (Eq.(33)) is represented in the right panel. Each arrow indicates the wind speed and direction for a set of equidistant values of the flux variable a. The hatched region is the boundary layer about the polar axis.